Character of the Symmetric Group Action on the Cohomology of the Pure Virtual and Flat Braid Groups

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Abstract

In this paper we give Hilbert series for the character of the action of the symmetric groups S_n on the cohomology algebras of the groups PvB_n (pure virtual braid groups) and PfB_n (pure flat braid groups), and on their quadratic dual algebras.

Contents

1 Introduction

In this paper we will be concerned with the action of the symmetric groups S_n on the cohomology algebras of the groups PvB_n (pure virtual braid groups) and PfB_n (pure flat braid groups), and on their quadratic dual algebras.

There has been, and continues to be, a great deal of research into the action of S_n on the cohomology algebra of the pure braid group PB_n, which is closely related to PvB_n and PfB_n, and more generally on the action of Coxeter groups on the corresponding generalized pure braid groups: see, for instance, [d'A-G], [D-P-R], [F-V], [L], [L-S]. In particular, Lehrer and Solomon [L-S] gave a formula for the (non-graded) character of S_n on the total cohomology of PB_n. Subsequently, Lehrer [L], and Blair and Lehrer [B-L], gave Hilbert series describing the graded character of the action of Coxeter groups on the cohomology of the complements of the related hyperplanes; when the Coxeter group is S_n, this is equivalent to the graded charcter of S_n on the cohomology of PB_n.

The interest in the S_n-action on the cohomology of PvB_n and PfB_n derives partly from these groups' close relation to PB_n. In particular, there is an 'almost' exact sequence of groups:

$$0 \to PB_n \xrightarrow{\Psi_n} PvB_n \xrightarrow{\Pi_n} PfB_n \longrightarrow 0$$

This sequence is exact on the left and right, and moreover the kernel of Π_n is the normal closure of the image of Ψ_n.

PvB_n was studied by Bartholdi, Enriques, Etingof and Rains in [BEER][1] as the group generated by symbols R_{ij}, $1 \leq i \neq j \leq n$, subject to the Yang-Baxter (or Reidemeister III) relations and certain commutativities:

$$R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij} \tag{1}$$
$$R_{ij}R_{kl} = R_{kl}R_{ij}, \tag{2}$$

with i, j, k, l distinct. The groups PfB_n are given by the same presentation but subject to the additional relations $R_{ij}R_{ji} = 1$.

A further reason for the interest in PvB_n, particularly, is its relation with the pure string motion group PwB_n (also known as the pure welded braid group, the McCool group or the group of pure symmetric automorphisms of a free group of rank n). PwB_n is the quotient of PvB_n by the further relations $R_{ij}R_{ik} = R_{ik}R_{ij}$, when i, j, k are distinct.

If we put $G = PvB_n$ or $G = PfB_n$, the group ring $\mathbb{Q}G$ has a natural augmentation $\mathbb{Q}G \twoheadrightarrow \mathbb{Q}$ defined by sending each generator to $1 \in \mathbb{Q}$. After filtering the group ring $\mathbb{Q}G$ by powers of the augmentation ideal (that is, the kernel of the above augmentation map), the associated graded rings $gr\mathbb{Q}G$ are known as $\mathfrak{pvb}_n$ and $\mathfrak{pfb}_n$ (for $G = PvB_n$ and $G = PfB_n$, respectively). Their 'quadratic dual' algebras $\mathfrak{pvb}_n^!$ and $\mathfrak{pfb}_n^!$ are known to coincide with the cohomology algebras of the corresponding groups (see [BEER] and [Lee]):

[1] In [BEER], PvB_n is referred to as the quasi-triangular group QTr_n, while PfB_n is referred to as the triangular group Tr_n.

$$\mathfrak{pvb}_n^! \cong H^*(PvB_n, \mathbb{Q})$$
$$\mathfrak{pfb}_n^! \cong H^*(PfB_n, \mathbb{Q})$$

There is an action of the symmetric group S_n on the graded components $\mathfrak{pvb}_n^k$ and $\mathfrak{pvb}_n^{!k}$ of $\mathfrak{pvb}_n$ and $\mathfrak{pvb}_n^!$, and similarly on the graded components $\mathfrak{pfb}_n^k$ and $\mathfrak{pfb}_n^{!k}$ of $\mathfrak{pfb}_n$ and $\mathfrak{pfb}_n^!$.

In this paper we give Hilbert series encoding the characters of these actions.

Our results for $\mathfrak{pvb}_n$ and $\mathfrak{pfb}_n$ are given in terms of the graded characters for $\mathfrak{pvb}_n^!$ and $\mathfrak{pfb}_n^!$, respectively, by means of a 'Koszul formula' for graded characters, which extends the standard Koszul formula which relates the Hilbert series encoding the graded dimensions of a Koszul algebra with the Hilbert series of the algebra's quadratic dual: see Theorem 4.

We now give a more precise statement of our principal results. We will denote by $\mathfrak{pvb}_{n,\sigma}^k$ and $\mathfrak{pvb}_{n,\sigma}^{!k}$ the character of the S_n-action on the graded components $\mathfrak{pvb}_n^k$ and $\mathfrak{pvb}_n^{!k}$, evaluated at any $\sigma \in S_n$. We define:

$$\mathfrak{pvb}_{n,\sigma}^!(z) := \sum_{k \geq 0} \mathfrak{pvb}_{n,\sigma}^{!k} z^k$$

and similarly for $\mathfrak{pvb}_{n,\sigma}(z)$. We take $\mathfrak{pvb}_1 = \mathfrak{pvb}_1^! = \mathbb{Q}$ to be the trivial representation, so that $\mathfrak{pvb}_1(z) = \mathfrak{pvb}_1^!(z) = 1$. We use similar notation for the character of S_n on $\mathfrak{pfb}_n$ and $\mathfrak{pfb}_n^!$.

In Theorem 2 we prove that, if $\sigma \in S_n$ has cycle type corresponding to a 'homogeneous' partition of n, that is $n = k + \cdots + k$ (with α_k summands), then:

$$\mathfrak{pvb}_{n,\sigma}^!(z) = \sum_{0 \leq \beta \leq \alpha_k} L(\alpha_k, \beta)(-1)^{(\alpha_k - \beta)(k-1)} k^{(\alpha_k - \beta)} z^{(\alpha_k - \beta)k}$$

where the $L(p,q)$ stand for Lah numbers, which count the number of unordered partitions of $[p] := \{1, \ldots, p\}$ into q ordered subsets. This extends the case $\sigma = 1 \in S_n$, which gives the Hilbert series for the graded dimensions of $\mathfrak{pvb}_n^!$, which were derived in [BEER].

Moreover, if $\sigma \in S_n$ has cycle type corresponding to a non-homogeneous partition $n = \sum_{i=1}^r i\alpha_i$, with $i, \alpha_i, r \in \mathbb{N}$; and if we define $n_i = i\alpha_i$, for $i = 1, \ldots, r$, and denote $\mathfrak{pvb}_{n_i, i\alpha_i}^!$ the character (given above) corresponding to the homogeneous partition $n_i = i + \cdots + i$ (α_i summands), then:

$$\mathfrak{pvb}_{n,\sigma}^!(z) = \prod_{i=1}^r \mathfrak{pvb}_{n_i, i\alpha_i}^!$$

In Corollary 1 the Koszul formula for graded characters (mentioned above) applies to give:

$$\mathfrak{pvb}_{n,\sigma}(z) = \frac{1}{\mathfrak{pvq}_{n,\sigma}^!(-z)}$$

The graded characters for $\mathfrak{pfb}_n^!$ and $\mathfrak{pfb}_n$ are given in Theorem 3 and Corollary 4.

The paper is organized as follows. In Part 2 we review some background concerning the groups PvB_n and PfB_n, including their relations to the pure braid groups PB_n. We further review the definition of the associated graded algebras $\mathfrak{pvb}_n$ and $\mathfrak{pfb}_n$, and their quadratic duals (which correspond to the cohomology algebras of the respective groups). We also explain the action of S_n on these algebras. In Part 3 we show that the representation of S_n on the top degree component of $\mathfrak{pvb}_n^!$ is in fact the regular representation. We then state and prove formulas for the character of S_n on $\mathfrak{pvb}_n^!$ and $\mathfrak{pvb}_n$. In Part 4 we show that the representation of S_n on the top degree component of $\mathfrak{pfb}_n^!$ is the alternating representation. We then state and prove formulas for the character of S_n on $\mathfrak{pfb}_n^!$ and $\mathfrak{pfb}_n$. In Part 5 we state and prove the extended Koszul formula for characters of a finite group on a quadratic algebra. In Part 6 we make some final comments.

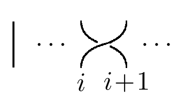
2 Background Concerning the Pure Virtual Braid Groups

As the names suggest, the pure virtual braid groups and the pure flat braid groups are closely related to the pure braid groups. We explain these relationships here, following [BND], and [Lee] (Section 2.4).

Recall that the braid group B_n is generated by the symbols $\{\sigma_i : i = 1, \ldots, (n-1)\}$, subject to the Reidemeister III relation and obvious commutativities:

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \qquad \text{for } |i-j|>1 \end{aligned}$$

The generator σ_i may be interpreted as corresponding to a braid with n strands with the strand in position i crossing over the adjacent strand to the 'right' (i.e. the strand in position $(i+1)$), in a 'positive' fashion:



(we draw all strands with upwards orientation).

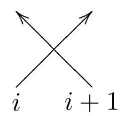
We obtain the (non-pure) *virtual* braid group vB_n by adding to B_n the generators $\{s_i : i = 1, \ldots, (n-1)\}$, referred to as virtual crossings. The $\{s_i\}$ are subject to the symmetric group relations:

$$\begin{aligned} s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} \\ s_is_j &= s_js_i \qquad \text{for } |i-j|>1 \\ s_i^2 &= 1 \end{aligned}$$

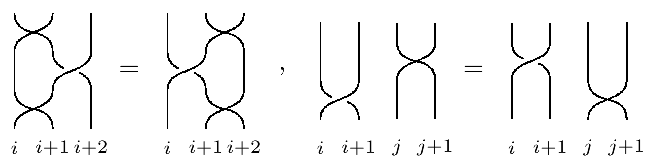
The σ_i and the s_i are subject to certain 'mixed' relations:

$$\begin{aligned} s_i\sigma_{i+1}^{\pm1}s_i &= s_{i+1}\sigma_i^{\pm1}s_{i+1} \\ s_i\sigma_j &= \sigma_j s_i \qquad \text{for } |i-j|>1 \end{aligned}$$

In pictures, virtual crossings s_i are drawn as



Hence the pictures corresponding to the mixed relations are as follows (for the case where all ordinary crossings are positive – one can draw similar pictures when the ordinary crossings are negative):

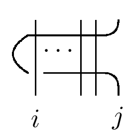


One can show that the $\{\sigma_i\}$ generate a copy of the braid group B_n, while the $\{s_i\}$ generate a copy of the symmetric group S_n, within vB_n. The map which sends each σ_i and s_i to s_i gives a surjection $vB_n \twoheadrightarrow S_n$,whose kernel is by definition the pure virtual braid group, PvB_n. A presentation for PvB_n may be obtained using the Reidemeister-Schreier method, and the reader is referred to [Bard] for a through explanation, or to [Lee] (Section 2.4) for a quick overview.

One finds that PvB_n is generated by the set $\{R_{ij}\}_{1\le i\ne j\le n}$, subject to the relations:

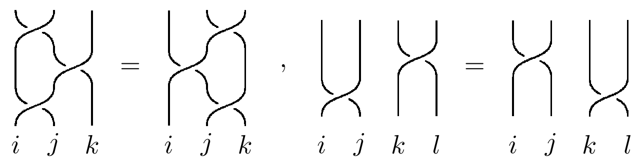
$$\begin{aligned} R_{ij}R_{ik}R_{jk} &= R_{jk}R_{ik}R_{ij} \\ R_{ij}R_{kl} &= R_{kl}R_{ij}, \end{aligned}$$

with i, j, k, l distinct. A typical element R_{ij} may be depicted:



(with all strands oriented upwards). One can show that the relations satisfied by the $\{s_i\}$ (mixed and unmixed) ensure that one can think of each generator R_{ij} as an ordinary (positive) crossing of strand i over strand j, with an arbitrary choice of virtual moves before and after the ordinary crossings to get the strands 'into position' (see [BND]). Hence in pictures one does not usually show the virtual crossings.

The relations in the pure virtual braid group may be illustrated as follows:



PfB_n has the same presentation as PvB_n, but subject to the additional relations $R_{ij}R_{ji} = 1$.

In a similar way the pure braid group PB_n is, by definition, the kernel of the group homomorphism $B_n \twoheadrightarrow S_n$ which sends each σ_i to s_i. PB_n is generated by the symbols $\{A_{ij} : 1 \leq i < j \leq n\}$ (for the relations, see e.g. [MarMc]). As pointed out in [BEER] (Section 4.3), there is a homomorphism $\Psi_n : PB_n \to PvB_n$ defined by:

$$A_{ij} \mapsto R_{j-1,j} \dots R_{i+1,j} R_{ij} R_{ji} (R_{j-1,j} \dots R_{i+1,j})^{-1}$$

There are also natural homomorphisms $PfB_n \to PvB_n \to PfB_n$, with the composition being the identity (this observation is due to [BEER], Section 2.3). The second map, Π_n, sends all generators R_{ij} to themselves, and the first sends R_{ij} to itself whenever $i < j$. We conclude that that PfB_n is a split quotient of PvB_n.

One obtains a complex:

$$0 \to PB_n \xrightarrow{\Psi_n} PvB_n \xrightarrow{\Pi_n} PfB_n \longrightarrow 0$$

which is clearly exact on the right, and is shown to be exact on the left in [BEER]; moreover, the kernel of Π_n is readily seen to be the normal closure of the image of Ψ_n.

2.1 Associated Graded Algebras of the Group Algebras

For any finitely presented group G, the group ring $\mathbb{Q}G$ has a natural augmentation $\mathbb{Q}G \twoheadrightarrow \mathbb{Q}$ defined by sending each generator to $1 \in \mathbb{Q}$. If one filters the group ring $\mathbb{Q}G$ by powers of the augmentation ideal (that is, the kernel of the above augmentation map), the associated graded ring $gr\mathbb{Q}G$ becomes a natural object of study.

In the case of PB_n, the associated graded $\mathfrak{pb}_n := gr\mathbb{Q}PB_n$ (often referred to as the chord diagram algebra) is known[2] to be generated by symbols $\{a_{ij} : 1 \leq i \neq j \leq n\}$, subject to the relations:

$$[a_{ij}, a_{ik} + a_{jk}] = 0$$
$$[a_{ij}, a_{kl}] = 0$$

for distinct i, j, k, l.

[2] See for instance [Hut] and [Koh].

In the case of PvB_n, the associated graded $\mathfrak{pvb}_n := gr\mathbb{Q}PvB_n$ is known[3] to be generated by symbols $\{r_{ij} : 1 \leq i \neq j \leq n\}$, subject to the relations:

$$[r_{ij}, r_{ik} + r_{jk}] + [r_{ik}, r_{jk}] = 0$$
$$[r_{ij}, r_{kl}] = 0$$

for distinct i, j, k, l.

Finally, the associated graded $\mathfrak{pfb}_n := gr\mathbb{Q}PfB_n$ is known[4] to have the same presentation as $\mathfrak{pvb}_n$, but with the extra relations $r_{ij} + r_{ji} = 0$.

As in the case of the corresponding groups, one has homomorphisms $\psi_n : \mathfrak{pb}_n \hookrightarrow \mathfrak{pvb}_n$ defined by $a_{ij} \mapsto r_{ij} + r_{ji}$, and $\pi_n : \mathfrak{pvb}_n \twoheadrightarrow \mathfrak{pfb}_n$ which is just the quotient map. Then again one has a complex

$$0 \to \mathfrak{pb}_n \xrightarrow{\psi_n} \mathfrak{pvb}_n \xrightarrow{\pi_n} \mathfrak{pfb}_n \longrightarrow 0$$

which is exact on the right and left; moreover, the kernel of π_n is the normal closure of the image of ψ_n.[5]

2.2 The Cohomology Algebras of PB_n, PvB_n and PfB_n

We first recall the definition of quadratic algebras and their duals.

Let V be a finite-dimensional vector space, and denote TV the tensor algebra over V (we assume rational coefficients). Let $R \subseteq V \otimes V$ be a subspace and denote $\langle R \rangle$ the ideal in TV generated by R. These data permit one to define an algebra A as:

$$A := \frac{TV}{\langle R \rangle}$$

Such an algebra A is known as a quadratic algebra. The algebra A has a 'quadratic dual' $A^!$ algebra defined as follows. Let $R^\perp \subseteq V^* \otimes V^*$ be the annihilator of R. Then define:

$$A^! := \frac{TV^*}{\langle R^\perp \rangle}$$

The algebras $\mathfrak{pb}_n$, $\mathfrak{pvb}_n$ and $\mathfrak{pfb}_n$ are obviously quadratic. Their quadratic dual algebras $\mathfrak{pb}_n^!$, $\mathfrak{pvb}_n^!$ and $\mathfrak{pfb}_n^!$ are known to be the cohomology algebras of the corresponding groups:[6]

[3] See [BEER] and [Lee]. In the terminology of [BEER], $\mathfrak{pvb}_n$ is $U(\mathfrak{qtr}_\mathfrak{n})$, the universal enveloping algebra of the 'quasi-triangular' Lie algebra $\mathfrak{qtr}_\mathfrak{n}$. The latter is the Lie algebra with the same generators and defining relations as $\mathfrak{pvb}_n$.

[4] See [BEER] and [Lee]. In the terminology of [BEER], $\mathfrak{pfb}_n$ is $U(\mathfrak{tr}_\mathfrak{n})$, the universal enveloping algebra of the 'triangular' Lie algebra $\mathfrak{tr}_\mathfrak{n}$, which in turn is the Lie algebra with the same generators and defining relations as $\mathfrak{pfb}_n$.

[5] These observations are due to [BEER].

[6] In the case of PB_n, see [Arn] and [Koh], and in the case of PvB_n and PfB_n see [BEER] and [Lee].

$$
\begin{aligned}
\mathfrak{pb}_n^! &\cong H^*(PB_n, \mathbb{Q}) \\
\mathfrak{pvb}_n^! &\cong H^*(PvB_n, \mathbb{Q}) \\
\mathfrak{pfb}_n^! &\cong H^*(PfB_n, \mathbb{Q})
\end{aligned}
$$

One can readily confirm that the algebra $\mathfrak{pvb}_n^!$ is the exterior algebra generated by the set $\{r_{ij} : 1 \le i \ne j \le n\}$[7], subject to the relations:

$$
\begin{aligned}
r_{ij} \wedge r_{ik} &= r_{ij} \wedge r_{jk} - r_{ik} \wedge r_{kj} \\
r_{ik} \wedge r_{jk} &= r_{ij} \wedge r_{jk} - r_{ji} \wedge r_{ik} \\
r_{ij} \wedge r_{ji} &= 0
\end{aligned}
\tag{3}
$$

where the indices i, j, k are all distinct.

In a similar way, one finds that the algebra $\mathfrak{pfb}_n^!$ is the exterior algebra generated by the set $\{r_{ij} : 1 \le i \ne j \le n,\ r_{ij} = -r_{ji}\}$ subject to the relations:

$$
\begin{aligned}
r_{ij} \wedge r_{ik} &= r_{ij} \wedge r_{jk} \\
r_{ik} \wedge r_{jk} &= r_{ij} \wedge r_{jk}
\end{aligned}
\tag{4}
$$

for all i, j, k such that $i < j < k$.

2.3 The Action of S_n

The symmetric group S_n acts on the generators $\{r_{ij}\}$ of $\mathfrak{pvb}_n$, $\mathfrak{pvb}_n^!$, $\mathfrak{pfb}_n$ and $\mathfrak{pfb}_n^!$ via the map $r_{ij} \mapsto r_{\sigma i, \sigma j}$ for $\sigma \in S_n$. It is easy to check that the relations in the respective algebras are respected by this action, so the action descends to the algebras themselves.

The following definition will be central in our computation of the characters of $\mathfrak{pvb}_n^!$:

Definition 1. *Let m be any monomial in a basis $\mathcal{B}$ for $\mathfrak{pvb}_n^!$. For any element $\sigma \in S_n$, let $\chi_\sigma(m)$ be the coefficient of m itself in the expansion of $\sigma(m)$ in terms of the basis $\mathcal{B}$. We say that m is a characteristic monomial for σ if the coefficient $\chi_\sigma(m)$ is non-zero.*

It is clear that:

$$
\mathfrak{pvb}_{n,\sigma}^!(z) = \sum_{\chi_\sigma(m) \ne 0} \chi_\sigma(m) z^{deg(m)}
$$

[7]Strictly speaking, the quadratic dual $\mathfrak{pvb}_n^!$ is generated by the dual generators $\{r_{ij}^*\}$. However to simplify the notation we will drop the stars. A similar convention will be adopted for $\mathfrak{pfb}_n^!$.

where the sum is over $m \in \mathcal{B}$ such that $\chi_\sigma(m) \neq 0$, and $deg(m)$ is the degree of m. The determination of the character of $\mathfrak{pvb}^!_n$ will be a matter of determining what are the characteristic monomials and counting their numbers and signs.[8]

We will adopt a similar definition and notation for the character of $\mathfrak{pfb}^!_n$.

3 Algebras Related to the Pure Virtual Braid Groups

3.1 Basis

Monomials in $\mathfrak{pvb}^!_n$ may conveniently be represented by graphs on the vertex set $[n] := \{1, \ldots, n\}$, with generators r_{ij} being represented by a directed edge or arrow from i to j. A given graph specifies a unique monomial up to sign.

In [BEER] it was shown that $\mathfrak{pvb}^!_n$ is Koszul and has the Hilbert series:

$$\mathfrak{pvb}^!_n(z) = \sum_{i=0}^{n} L(n,i) z^{(n-i)} \tag{5}$$

where the $L(n,i)$ stand for Lah numbers (counting the number of unordered partitions of $[n]$ into i ordered subsets). In particular, $dim\ \mathfrak{pvb}^{!n-1}_n = n!$.

In [Lee] a basis is given for $\mathfrak{pvb}^!_n$ which makes clear the dependance on Lah numbers, and we recall this now. A monomial in $\mathfrak{pvb}^!_n$ (and its corresponding graph) is called admissible if the monomial is of the form $r_{i_1 i_2} \wedge r_{i_2 i_3} \wedge \ldots \wedge r_{i_{m-1} i_m}$ with distinct i_l, $1 \leq i_l \leq n$. The set $\{i_1, i_2, \ldots, i_m\}$ is called the support of the monomial, and i_1 the root[9]. Thus admissible graphs are oriented chains of the form:

$$\overset{i_1}{\longrightarrow}\overset{i_2}{\cdots\cdots\cdots}\overset{i_m}{\longrightarrow}$$

The following is Theorem 7 from [Lee] (in slightly different language):

Proposition 1. *Products of admissible monomials with disjoint supports (in the order of increasing roots) form a basis $\mathcal{B}$ for $\mathfrak{pvb}^!_n$.*

The following lemma is straightforward:

Lemma 1. *Any collection Γ of admissible graphs with disjoint supports determines a unique basis element in $\mathcal{B}$, namely the product of the corresponding monomials, ordered by increasing roots. If the union of the supports of Γ has cardinality α and Γ has β components, the degree of the basis element of PvB_n determined by Γ is $(\alpha - \beta)$.*

In light of the lemma, for notational simplicity we often conflate such a Γ and the basis element it determines.

[8] This overall manner of approach is similar to that pursued in [L], and is very effective in any case where a combinatorially workable basis for the representation is at hand.

[9] This terminology is borrowed from [BEER], where it was applied to a basis for $\mathfrak{pfb}^!_n$.

3.2 S_n Representation on Top Degree Component of $\mathfrak{pvb}^!_n$

We can easily determine the nature of the top-degree representation $\mathfrak{pvb}^{!n-1}_n$:

Theorem 1. *The top-degree representation $\mathfrak{pvb}^{!n-1}_n$ is (isomorphic to) the regular representation of S_n, for all n.*

The proof will be an immediate consequence of the following lemma:

Lemma 2. *The top-degree representation $\mathfrak{pvb}^{!n-1}_n$ has character $\mathfrak{pvb}^{!n-1}_{n,\sigma} = n!$, for $\sigma = 1$, and $\mathfrak{pvb}^{!n-1}_{n,\sigma} = 0$, for $\sigma \neq 1$.*

Proof. If Γ is a basis element and $\Gamma \in \mathfrak{pvb}^{!n-1}_n$, then Γ has just 1 connected component (see Lemma 1), which must be an admissible graph (that is, an oriented chain). Every permutation σ sends an oriented chain to another oriented chain $\sigma\Gamma$, which is therefore also an admissible graph. But then[10] we can only have $\chi_\sigma(\Gamma) = 1$ or $\chi_\sigma(\Gamma) = 0$, and the former holds if and only if $\sigma = 1$. Moreover, we know[11] that *dim* $\mathfrak{pvb}^{!n-1}_n = n!$, and the lemma follows. ☐

The Theorem is then an immediate consequence of the basic fact in the representation theory of the symmetric group that the character described in the lemma is that of the regular representation.

3.3 Graded Characters of $\mathfrak{pvb}^!_n$ and $\mathfrak{pvb}_n$

Theorem 2. *1). Let $\sigma \in S_n$ have cycle type corresponding to a 'homogeneous' partition of n, that is $n = k + \cdots + k$ (with α_k summands), for some $k, \alpha_k \geq 1$. Then:*

$$\mathfrak{pvb}^!_{n,\sigma}(z) = \mathfrak{pvb}^!_{\alpha_k}((-1)^{k-1} k z^k)$$

where $\mathfrak{pvb}^!_{\alpha_k}(z)$ is the Hilbert series for $\mathfrak{pvb}^!_{\alpha_k}$ given in (5).

2). Now let $\sigma \in S_n$ have cycle type corresponding to a non-homogeneous partition $n = \sum_{i=1}^r i\alpha_i$, with $i, \alpha_i, r \in \mathbb{N}$. Define $n_i = i\alpha_i$, for $i = 1, \ldots, r$, and denote $\mathfrak{pvb}^!_{n_i, i\alpha_i}$ the character (given in 1)) corresponding to the partition $n_i = i + \cdots + i$ (α_i summands). Then:

$$\mathfrak{pvb}^!_n(z) = \prod_{i=1}^r \mathfrak{pvb}^!_{n_i, i\alpha_i}$$

The following is an immediate consequence of the Theorem, using the 'Koszul formula' of Theorem 4:

Corollary 1. *The characters $\mathfrak{pvb}_{n,\sigma}(z)$ are given in terms of the $\mathfrak{pvb}^!_{n,\sigma}(z)$ by the Koszul formulas:*

$$\mathfrak{pvb}_{n,\sigma}(z) = \frac{1}{\mathfrak{pvq}^!_{n,\sigma}(-z)}$$

[10] See the definition of $\chi_\sigma(m)$ in Definition 1.
[11] See Equation (5) and the subsequent comments.

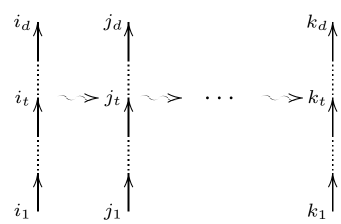
Since the character of a representation evaluated at a particular $\sigma \in S_n$ depends only on the conjugacy class to which σ belongs, we assume that σ may be presented as a product of disjoint cycles $\sigma_1\sigma_2\ldots$ such that each cycle is a list of consecutive (increasing) integers. In particular, each cycle of length i may be written $(a+1, a+2, \ldots, a+i)$, for some integer a.

Recall that in Definition 1, for any monomial m in the basis $\mathcal{B}$ for $\mathfrak{pvb}^!_n$, and for any element $\sigma \in S_n$, we defined $\chi_\sigma(m)$ to be the coefficient of m itself in the expansion of $\sigma(m)$ in terms of the basis $\mathcal{B}$; and we said that m is characteristic for σ if $\chi_\sigma(m) \neq 0$.

Proof of the Theorem. Let m be a characteristic monomial for σ, and in particular a basis element for $\mathfrak{pvb}^!_n$. Let Γ be the corresponding graph. We claim that $\sigma\Gamma$ is also a basis element (up to sign) without the need for any manipulation using the relations. Indeed, the connected components γ of Γ must be admissible graphs and have distinct supports, and it is clear (since σ is a bijection) that the $\sigma\gamma$ must also be admissible chains of $\sigma\Gamma$ with distinct supports, and the claim follows.

Let γ_1 be any chain of Γ. From the previous paragraph we conclude that $\sigma\Gamma = \pm\Gamma$. Hence either σ acts as the identity on γ_1 or $\sigma\gamma_1 = \gamma_2$ for some chain $\gamma_2 \neq \gamma_1$ of Γ. In either case, we see that in fact there must exist some integer $l \geq 1$ and distinct chains $\gamma_1, \ldots, \gamma_l$ in Γ such that $\sigma\gamma_1 = \gamma_2, \ldots, \sigma\gamma_l = \gamma_1$.

Suppose γ_1 (and hence $\gamma_j,\ j = 1, \ldots, l$) have length d. It is clear that for each $t = 1, \ldots, d$, the vertices in position t in $\gamma_1, \ldots, \gamma_l$ must comprise a single l-cycle in σ. (Note in particular that no two vertices from the same chain belong to the same cycle of σ.) The picture is as follows:



So we immediately get:

Proposition 2. *Let σ have cycle type corresponding to some partition $n = \sum_{1 \leq i \leq r} i\alpha_i$. Then Γ decomposes into components (not necessarily connected) $\Gamma_1, \ldots, \Gamma_r$ such that the i-cycles of σ permute the chains in Γ_i (not necessarily transitively) but are the identity on Γ_j for $j \neq i$.*

The following corollary is immediate:

Corollary 2.

$$\chi_\sigma(\Gamma) = \prod_{i=1}^{r} \chi_\sigma(\Gamma_i)$$

Thus we have reduced the problem to the case of σ corresponding to a homogeneous partition (with α_k cycles of size k, say). We continue to assume that the cycles in the cycle decomposition of σ (can be presented so as to) consist of consecutive integers, and we assume further that some ordering of the cycles has been fixed. Consider the vertex 1, which lies in the cycle $(12\dots k)$. We label the chain to which 1 belongs by γ_1. From the previous discussion, the vertices of γ_1 must consist of either 0 or 1 vertex from each cycle of σ. Once it is known which vertices lie in γ_1, then the vertices of $\gamma_2 := \sigma\gamma_1, \dots, \gamma_k := \sigma\gamma_{k-1}$ are determined.

If there are chains in Γ beyond $\gamma_1, \dots, \gamma_k$, we consider the remaining chain δ_1 with the smallest vertex; as before, the vertices of δ_1 must consist of either 0 or 1 vertex from each cycle of σ which is as yet unaccounted for. Once it is known which vertices lie in δ_1, there must be remaining chains $\delta_2, \dots, \delta_k$ in Γ, distinct from the γ_j, such that $\delta_2 := \sigma\delta_1, \dots, \delta_k := \sigma\delta_{k-1}$.

We repeat this process until all chains of Γ are accounted for. This leads us to the following proposition.

Proposition 3. *For every basis element μ in $\mathfrak{pvb}^!_{\alpha_k}$ (where α_k is the number of k-cycles in σ) with some number β of components, we get $k^{\alpha_k-\beta}$ distinct possible characteristic monomials Γ for σ, by replacing each vertex i of μ by a choice of any element in the i-th cycle of σ (thus producing a basis element ν in $\mathfrak{pvb}^!_n$), and then setting $\Gamma = \nu_1 \dots \nu_k$ with $\nu_i := \sigma^{(i-1)}\nu$.*

Moreover, all characteristic monomials of Γ arise this way.

Proof. Indeed, there are initially k^{α_k} ways to pick ν, but since cyclic relabelings of the $\nu_1 \dots \nu_k$ correspond to the same graph, we need to divide by a factor of k in respect of each component in ν. This results in $k^{\alpha_k-\beta}$ distinct choices. The fact that all characteristic monomials arise this way follows from the discussion immediately Proposition 2. □

Lemma 3. *Each characteristic monomial Γ constructed in Proposition 3 satisfies*

$$\sigma\Gamma = (-1)^{(\alpha_k-\beta)(k-1)}\Gamma$$

and has degree $(\alpha_k - \beta)k$.

Proof. Indeed (using the notation from Proposition 3) if μ has β components, then μ and ν have degree $(\alpha_k - \beta)$,[12] so Γ has degree $(\alpha_k - \beta)k$, as stated. Also,

$$\begin{aligned}\sigma\Gamma = \sigma\nu_1 \dots \sigma\nu_k = \nu_2 \dots \nu_k\nu_1 &= (-1)^{\sum_{j=2}^{k} |\nu_j||\nu_1|}\nu_1 \dots \nu_k \\ &= (-1)^{(\alpha_k-\beta)^2(k-1)}\nu_1 \dots \nu_k \\ &= (-1)^{(\alpha_k-\beta)(k-1)}\nu_1 \dots \nu_k\end{aligned}$$

(where we have conflated Γ and the ν_i with the basis elements that they determine, and thus view the above as an equation involving monomials in the anti-symmetric algebra $\mathfrak{pvb}^!_n$) as required. □

[12] See Lemma 1.

Recall from (5) that

$$\mathfrak{pvb}^!_{\alpha_k}(z) = \sum_{0\le\beta\le\alpha_k} L(\alpha_k,\beta) z^{(\alpha_k-\beta)}$$

From Proposition 3 and Lemma 3, we now see that

$$\mathfrak{pvb}^!_{n,\sigma}(z) = \sum_{0\le\beta\le\alpha_k} L(\alpha_k,\beta)(-1)^{(\alpha_k-\beta)(k-1)} k^{(\alpha_k-\beta)} z^{(\alpha_k-\beta)k} = \mathfrak{pvb}^!_{\alpha_k}((-1)^{(k-1)} k z^k)$$

This completes the proof of the theorem. □

4 Algebras Related to the Pure Flat Braid Groups

4.1 Basis

As with $\mathfrak{pvb}^!_n$, monomials in $\mathfrak{pfb}^!_n$ may be represented by graphs on the vertex set $[n] := \{1,\ldots,n\}$, with generators r_{ij} being represented by a directed edge or arrow from i to j, whenever $i<j$. Again a given graph specifies a unique monomial up to sign.

In [BEER] it was shown that $\mathfrak{pfb}^!_n$ is Koszul, by exhibiting a quadratic Gröbner basis for $\mathfrak{pfb}^!_n$ as an exterior algebra. They also gave a basis for $\mathfrak{pfb}^!_n$ itself.

We recall the terminology used in [BEER]. A monomial in $\mathfrak{pfb}^!_n$ is called reduced if it has the form $r_{i_1i_2}\wedge r_{i_1i_3}\wedge\ldots\wedge r_{i_1i_m}$ with $i_1<i_2<\cdots<i_m$. The set $\{i_1,i_2,\ldots,i_m\}$ is called the support of the monomial, and i_1 is its root. The following is Proposition 4.2 of [BEER]:

Proposition 4. *Products of reduced monomials with disjoint supports (in the order of increasing roots) form a basis for* $\mathfrak{pfb}^!_n$.

We will use a related but slightly different basis for $\mathfrak{pfb}^!_n$. Specifically, we will say that a monomial whose graph is a directed chain with indices increasing in the direction of the arrows, is an admissible monomial (and the related graph is an admissible graph). In terms of the generators, these monomials have the form $r_{i_1i_2}\wedge r_{i_2i_3}\wedge\ldots\wedge r_{i_{m-1}i_m}$ with $i_1<i_2<\cdots<i_m$. The set $\{i_1,i_2,\ldots,i_m\}$ is again called the support of the monomial, and i_1 the root. It is easy to see that each subset of $[n]$ determines exactly one reduced monomial and one admissible monomial, and that these are equal modulo the relations, up to sign (we consider that singletons and the empty set determine $1\in\mathfrak{pvb}^!_n$). We therefore have:

Proposition 5. *Products of admissible monomials with disjoint supports (in the order of increasing roots) form a basis* $\mathcal{B}$ *for* $\mathfrak{pfb}^!_n$.

We will find it convenient to use the basis induced from admissible monomials, rather than reduced monomials.

Lemma 1, originally stated for $\mathfrak{pvb}^!_n$, still applies for $\mathfrak{pfb}^!_n$, and we repeat (and slightly extend) it here for convenience.

Lemma 4. *Any collection Γ of admissible graphs with disjoint supports determines a unique basis element in $\mathcal{B}$, namely the product of the corresponding monomials, ordered by increasing roots. If the union of the supports of Γ has cardinality α and Γ has β components, the degree of the basis element of $\mathfrak{pfb}_n^!$ determined by Γ is $(\alpha - \beta)$.*

It follows, in particular, that each partition of $[n]$ determines a unique (and distinct) basis element of $\mathcal{B}$, since each subset of $[n]$ determines a unique admissible chain.

In light of the lemma, for notational simplicity we often conflate such a Γ and the basis element it determines.

4.2 S_n Representation on Top Degree Component of $\mathfrak{pfb}_n^!$

We can very easily determine the S_n representation given by the top degree component of $\mathfrak{pfb}_n^!$. Indeed, this component is generated by the unique admissible monomial on the full set $[n]$. It therefore has degree $(n-1)$ and dimension 1. In fact:

Proposition 6. *The top degree component $\mathfrak{pfb}_n^{!(n-1)}$ of $\mathfrak{pfb}_n^!$ is the alternating representation of S_n.*

Proof. Indeed, the element (12) of S_n acts as follows, if $n > 2$:

$$\begin{aligned}(12)\ r_{12} \wedge r_{23} \wedge \ldots \wedge r_{(m-1)m} &= r_{21} \wedge r_{13} \wedge \ldots \wedge r_{(m-1)m} \\ &= -r_{12} \wedge r_{23} \wedge \ldots \wedge r_{(m-1)m}\end{aligned}$$

and (12) $r_{12} = -r_{12}$ if $n = 2$. Thus $\mathfrak{pfb}_n^{!(n-1)}$ is not the trivial representation, and so (being 1-dimensional) must be the alternating representation. □

The following is an easy corollary:

Corollary 3. *Let $S \subseteq [n]$ and let γ be the admissible chain in $\mathfrak{pvb}_n^!$ on the set S. Let $T \subseteq S$ be any subset and τ be a permutation of T. Then:*

$$\tau\gamma = sgn(\tau)\gamma$$

where $sgn(\tau)$ is the sign of τ.

4.3 Graded Characters of $\mathfrak{pfb}_n^!$ and $\mathfrak{pfb}_n$

In [BEER] it was shown that the Hilbert Series for $\mathfrak{pfb}_n^!$ is

$$\mathfrak{pfb}_n^!(z) = \sum_{0 \le k \le n} S(n, n-k) z^k \tag{6}$$

where the $S(n,k)$ are the Stirling numbers of the second kind, which give the number of (unordered) partitions of $[n]$ into k (unordered) subsets. The character formula that follows generalizes that result. We let $V(\Gamma)$ denote the set

of vertices of a graph Γ, and let $V(\tau) = S$ denote the set of indices of any permutation τ.

We take $\sigma \in S_n$, and as with the case of $\mathfrak{pvb}^!_n$ we assume that σ has a presentation as a product of disjoint cycles where each cycle may be written as a list of increasing consecutive integers.

Theorem 3. *The character $\mathfrak{pfb}^!_{n,\sigma}(z)$ is given by:*

$$\mathfrak{pfb}^!_{n,\sigma}(z) = \sum_S \prod_{i=1}^r \big[\sum_{k_i} \epsilon_{k_i} k_i^{(|S_i|-1)} z^{k_i(\sum_{\tau \in S_i} d_\tau - 1)} \big]$$

where

- *the first sum is over the unordered partitions $S = S_1 \sqcup \cdots \sqcup S_r$ of the set of cycles in the cycle decomposition of σ into unordered subsets;*
- *the second sum is over $k_i \geq 1$ which divide the orders of all cycles which belong to S_i; that is, such that there exist $d_\tau \geq 1$ satisfying $k_i d_\tau = |V(\tau)|$ for all cycles τ in S_i; and*
- $\epsilon_{k_i} = (-1)^{(k_i-1)(\sum_{\tau \in S_i} d_\tau - 1) + \sum_{\tau \in S_i}(d_\tau - 1)}$.

Note that the formula (6) is the case $\sigma = 1 \in S_n$; indeed, all cycles in $\sigma = 1$ have length 1, so the k_i and d_τ are all 1, and then $\epsilon_{k_i} = +1$.

Corollary 4. *The characters $\mathfrak{pfb}_{n,\sigma}(z)$ are given, in terms of the $\mathfrak{pfb}^!_{n,\sigma}(z)$, by the Koszul formulas:*

$$\mathfrak{pfb}_{n,\sigma}(z) = \frac{1}{\mathfrak{pfb}^!_{n,\sigma}(-z)}$$

As with $\mathfrak{pvb}^!_n$, for any element $\sigma \in S_n$ and any monomial in the basis $\mathcal{B}$ for $\mathfrak{pfb}^!_n$ set out in Proposition 5, we let $\chi_\sigma(m)$ be the coefficient of m itself in the expansion of $\sigma(m)$ in terms of the basis $\mathcal{B}$; and we say that m is a characteristic monomial for σ if the coefficient $\chi_\sigma(m)$ is non-zero (see Definition 1). The proof of the theorem is again a matter of determining what are the characteristic monomials and counting their numbers and signs.

Proof of Theorem 3. Since the defining relations in $\mathfrak{pfb}^!_n$ are binomial expressions with coefficients ± 1, the expansion of $\sigma(m)$ in terms of $\mathcal{B}$ will have a single non-zero term and the only questions are whether $\chi_\sigma(m) \neq 0$ and if so what is the sign. The following is the key proposition in that regard.

Proposition 7. *Let $\sigma \in S_n$ and suppose that every cycle τ in the cycle decomposition of σ consists of consecutive integers, that is $\tau = (\alpha_\tau + 1, \alpha_\tau + 2, \ldots, \alpha_\tau + |V(\tau)|)$, for some natural number α_τ. Suppose $m \in \mathcal{B}$ satisfies $\chi_\sigma(m) \neq 0$, and let Γ be the corresponding graph. Then there exist:*

1. *a unique unordered partition $S = S_1 \sqcup \cdots \sqcup S_r$ of the set of cycles in the cycle decomposition of σ (Note: the cycles within any particular S_i need not have the same length); and*

2. *unique integers k_i, $i = 1, \ldots, r$, such that k_i divides the orders of all cycles which belong to S_i; that is, such that there exist $d_\tau \geq 1$ satisfying $k_i d_\tau = |V(\tau)|$ for all cycles τ in S_i; and*

3. *for each $i = 1, \ldots, r$ and $\tau \in S_i$, with $\tau = (\alpha_\tau + 1, \alpha_\tau + 2, \ldots, \alpha_\tau + |V(\tau)|)$, a unique $t_\tau \in \{\alpha_\tau + 1, \alpha_\tau + 2, \ldots, \alpha_\tau + k_i)$*

such that:

A *Γ consists of r components $\Gamma_1 \ \ldots \ \Gamma_r$ (not necessarily connected);*

B *each Γ_i consists of k_i connected components, $\Gamma_i = \gamma_1 \ \ldots \ \gamma_{k_i}$, unique up to cyclic relabeling of the γ_j;*

C *we have $V(\gamma_1) = \bigcup_{\tau \in S_i} \{t_\tau + l k_i : l = 0, \ldots, (d_\tau - 1)\}$ (and γ_1 is the unique admissible graph on that collection of indices); and*

D *$\gamma_j = \sigma^{(j-1)} \gamma_1 \quad j = 1, \ldots, k_i$.*

Example 1.

We illustrate the above proposition by taking the case of

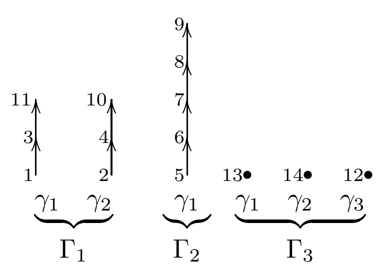
$$\sigma = (1234)(5)(6789)(10\ 11)(12\ 13\ 14)$$

Suppose we partition the cycles in σ into

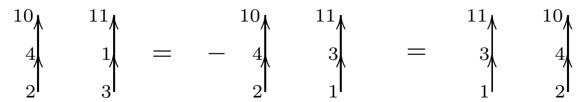
$$S = \{(1234), (10\ 11)\} \sqcup \{(5), (6789)\} \sqcup \{(12\ 13\ 14)\}$$

We take $k_1 = 2, k_2 = 1, k_3 = 3$ (noting that these do, indeed, divide the orders of the cycles in parts 1,2 and 3, respectively, of S). Finally we take $t_{(1234)} = 1, t_{(10\ 11)} = 11, t_{(5)} = 5, t_{(6789)} = 6$ and $t_{(12\ 13\ 14)} = 13$.

Then the graph which is determined by these data is the following:



One sees in particular that the components within each Γ_i are cyclically permuted (with signs). Specifically,

$\sigma(\Gamma_1) =$ 

Similarly one finds that $\sigma(\Gamma_2) = -\Gamma_2$, and

$$\sigma(\Gamma_3) = \;{}_{14}\bullet \quad {}_{12}\bullet \quad {}_{13}\bullet \;=\; \Gamma_3$$

Proof of Proposition 7. Since $\sigma(\Gamma) = \pm\Gamma$, it must be possible to group the connected components of Γ into collections $\Gamma_1, \dots, \Gamma_r$ such that the connected components within each Γ_i are cyclically permuted by σ.

For any one of these collections Γ_i, let $\gamma_1, \dots, \gamma_{k_i}$ be the connected components of Γ_i, labeled so that $\sigma(\gamma_1) = \gamma_2, \dots, \sigma(\gamma_{k_i}) = \gamma_1$. Obviously, the numbering of the k_i connected components is determined precisely up to cyclic relabeling of the γ_j.

Let τ be a cycle in the cycle decomposition of σ such that $V(\tau) \cap V(\Gamma_i) \neq \emptyset$. Then we must have

$$\tau\big(V(\tau) \cap V(\gamma_1)\big) = V(\tau) \cap V(\gamma_2), \quad \dots, \quad \tau\big(V(\tau) \cap V(\gamma_{k_i})\big) = V(\tau) \cap V(\gamma_1)$$

Hence $V(\tau) \subseteq V(\Gamma_i)$. Moreover, the $V(\tau) \cap V(\gamma_j)$ must all have the same size, which we call d_τ. Thus

$$|V(\tau)| = k_i d_\tau$$

and in particular k_i divides $|V(\tau)|$ for every cycle τ in σ such that $V(\tau) \cap V(\Gamma_i) \neq \emptyset$.

Let t_τ be the smallest element of $V(\tau) \cap V(\gamma_1)$. Then, recalling that we have assumed that τ (may be presented so that it) is a list of consecutive integers, we must have

$$t_\tau + l \in V(\tau) \cap V(\gamma_{l+1}), \quad \forall\, l = 0, \dots, (k_i - 1)$$

and then $t_\tau + k_i \in V(\gamma_1)$. Similarly, we find that

$$t_\tau + k_i + l \in V(\tau) \cap V(\gamma_{l+1}), \quad \forall\, l = 0, \dots, (k_i - 1)$$

and then $t_\tau + 2k_i \in V(\gamma_1)$. Continuing in this way, we conclude that

$$V(\tau) \cap V(\gamma_1) = \{t_\tau + sk_i,\ s = 0, \dots, d_\tau - 1\} \tag{7}$$

and

$$V(\tau) \cap V(\gamma_j) = \sigma^{(j-1)}\big(V(\tau) \cap V(\gamma_1)\big), \quad j = 1, \dots, k_i$$

Thus we see that

$$V(\gamma_1) = \bigcup_{\tau \in S_i} \{t_\tau + sk_i : s = 0, \ldots, (d_\tau - 1)\}$$

(and since Γ is a basis element, γ_1 must be the unique admissible graph on that set). Furthermore,

$$\gamma_j = \sigma^{(j-1)}\gamma_1 \quad j = 1, \ldots, k_i$$

Moreover, since t_τ is the smallest element of $V(\tau) \cap V(\gamma_1)$, and $V(\tau) = \{\alpha_\tau + 1, \ldots, \alpha_\tau + k_i d_\tau\}$, for some natural number α_τ, we conclude from (7) that $t_\tau \in \{\alpha_\tau + 1, \ldots, \alpha_\tau + k_i\}$.

As noted previously, if τ is a cycle in σ such that $V(\tau) \cap V(\Gamma_i) \neq \emptyset$, then in fact $V(\tau) \subseteq V(\Gamma_i)$. Thus each collection Γ_i determines a unique subset of the cycles in σ, and so the collection $S = S_1 \sqcup \cdots \sqcup S_r$ of the $\{\Gamma_i\}$ determines a unique partition of the cycles in σ.

We have thus seen how each characteristic monomial determines uniquely the data described in the proposition, as required. □

Proposition 8. *For any $\sigma \in S_n$, each possible choice of data as per 1-3 of Proposition 7, that is:*

- *an unordered partition $S = S_1 \sqcup \cdots \sqcup S_r$ of the cycles in the cycle decomposition of σ;*
- *integers k_i, $i = 1, \ldots, r$, such that k_i divides the orders of all cycles which belong to S_i; that is, such that there exist $d_\tau \geq 1$ satisfying $k_i d_\tau = |V(\tau)|$ for all cycles τ in S_i; and*
- *for each $i = 1, \ldots, r$ and $\tau \in S_i$, with $\tau = (\alpha_\tau + 1, \alpha_\tau + 2, \ldots, \alpha_\tau + |V(\tau)|)$, some $t_\tau \in \{\alpha_\tau + 1, \alpha_\tau + 2, \ldots, \alpha_\tau + k_i)$*

gives rise to a unique characteristic monomial of the form described in A-D of that proposition, and these are all distinct (up to cyclic relabelings of the γ_j within each $\Gamma_i, i = 1, \ldots, r$).

Proof. It is fairly clear that given the data 1-3 we can form a unique graph Γ as per A-D of Proposition 7. Moreover, these are distinct, up to cyclic relabelings of the $\gamma_j, j = 1, \ldots, k_i$ within each Γ_i. Indeed the vertex sets $V(\gamma)$ of the connected components γ of Γ determine a partition of $[n]$, and different Γ determine different partitions. The claim then follows because Lemma 4 implies that each partition determines a unique and distinct basis element.

By construction, σ just permutes the $\gamma_j, j = 1, \ldots, k_i$, within each Γ_i, so that $\sigma(\Gamma) = \pm\Gamma$ and Γ is a characteristic monomial for σ.

□

Proposition 9. *For any $\sigma \in S_n$, and for each possible choice of data as per Proposition 7, each component Γ_i of the resulting graph satisfies*

$$\sigma(\Gamma_i) = (-1)^{(k_i-1)(\sum_\tau d_\tau - 1) + \sum_\tau (d_\tau - 1)}\ \Gamma_i$$

and has degree $k_i(\sum_\tau d_\tau - 1)$. Furthermore, there are exactly $k_i^{(|S_i|-1)}$ choices of the $t_\tau \in \{\alpha_\tau+1, \alpha_\tau+2, \ldots, \alpha_\tau+k_i\}$, after factoring out cyclic relabelings of the γ_j.

Hence the space of characteristic monomials corresponding to any particular partition S of the cycles in the cycle decomposition of σ has character

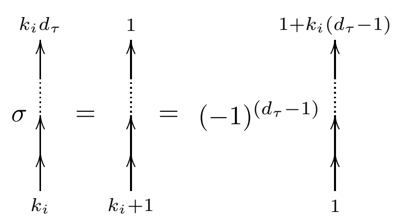
$$\chi_S = \prod_{S_i \in S} \sum_{k_i} (-1)^{(k_i-1)(\sum_\tau d_\tau - 1) + \sum_\tau (d_\tau - 1)} k_i^{(|S_i|-1)} z^{k_i(\sum_\tau d_\tau - 1)}$$

where the sum is over all k_i dividing $|V(\tau)|$ for each cycle τ in σ such that $V(\tau) \cap V(\Gamma_i) \neq \emptyset$.

Proof. We first determine the degree of the monomials corresponding to each Γ_i, $i = 1, \ldots, r$. Recall that for each connected component γ_j of Γ_i, the set $V(\gamma_j) \cap V(\tau)$ has d_τ elements. Hence $V(\gamma_j)$ has $\sum_\tau d_\tau$ elements (the sum being over all τ such that $V(\gamma_j) \cap V(\tau) \neq \emptyset$) and the degree of the monomial corresponding to γ_j is $\sum_\tau d_\tau - 1$. Hence the degree of the monomial corresponding to Γ_i is $k_i(\sum_\tau d_\tau - 1)$.

Let us now consider any particular Γ_i. Note that if σ is increasing on (the vertex set of) some admissible chain γ, then $\sigma\gamma$ remains an admissible graph. Also, for any cycle τ in σ such that $V(\tau) \subseteq V(\Gamma_i)$, τ is increasing on each set $V(\gamma_j) \cap V(\tau)$, except for the γ_j which contains the biggest element of τ, namely $\alpha_\tau + k_i d_\tau$. In fact τ is increasing even on this γ_j, except at $\alpha_\tau + k_i d_\tau$, which τ maps to $\alpha_\tau + 1$.

So we only need to apply relations to bring this particular $\sigma\gamma_j$ into admissible form. We recall that the indices in each τ are consecutive, in the sense that $\tau = (\alpha_\tau + 1, \ldots, \alpha_\tau + |V(\tau)|)$, for some α_τ; this implies that we can divide γ_j into disjoint segments, each corresponding to a different τ. Moreover, one need only reorder the indices within each segment (so that they are strictly increasing), and then the resulting graph will automatically be ordered. Thus it suffices to determine the required sign to reorder each segment, and then collect the signs. We illustrate this process in the following picture, for some segment corresponding to a particular τ, showing only the part of γ_j that involves indices from τ. For simplicity of notation we have assumed that $\alpha_\tau = 0$:



(for the sign, see Corollary 3.)

Hence in respect of each τ we get a sign $(-1)^{(d_\tau-1)}$, and so for Γ_i we get a sign $(-1)^{\sum_\tau (d_\tau-1)}$.

Next, recall that, by construction, $\sigma\Gamma_i = \pm\Gamma_i$. To determine what the sign is, note that

$$\begin{aligned}
\sigma(\gamma_1 \dots \gamma_{k_i}) &= (-1)^{\sum_\tau (d_\tau - 1)} \gamma_2 \dots \gamma_{k_i} \gamma_1 \\
&= (-1)^{\sum_\tau (d_\tau - 1)} (-1)^{(k_i - 1)|\gamma_1|^2} \gamma_1 \dots \gamma_{k_i} \\
&= (-1)^{\sum_\tau (d_\tau - 1)} (-1)^{(k_i - 1)|\gamma_1|} \gamma_1 \dots \gamma_{k_i} \\
&= (-1)^{\sum_\tau (d_\tau - 1) + (k_i - 1)(\sum_\tau d_\tau - 1)} \Gamma_i
\end{aligned}$$

Finally, after these steps $\sigma(\Gamma_i)$ has been brought into basis form, that is a product of admissible graphs with disjoint support and ordered by increasing roots, at the cost of the above signs.

For each part S_i of the partition S, and for each k_i dividing the orders of all the cycles τ such that $V(\tau) \subseteq V(\Gamma_i)$, there are $k_i^{|S_i|}$ ways to pick the $\{t_\tau\}$, but because the γ_j may be cyclically relabeled without changing the graph, we really only have $k_i^{(|S_i|-1)}$ possible graphs. Hence the character for the space of characteristic monomials corresponding to the part S_i is

$$\chi_{S_i} = \sum_{k_i} (-1)^{(k_i-1)(\sum_\tau d_\tau - 1) + \sum_\tau (d_\tau - 1)} k_i^{(|S_i|-1)} z^{k_i(\sum_\tau d_\tau - 1)}$$

where, as usual, the sum is over k_i dividing the orders of all the cycles τ such that $V(\tau) \subseteq V(\Gamma_i)$; and, for each k_i and each such τ, $|V(\tau)| = k_i d_\tau$.

We note that the space of characteristic monomials for σ induced by the partition S is isomorphic to the tensor product of the spaces of characteristic monomials for each S_i, so that

$$\chi_S = \prod_i \chi_{S_i}$$

$\square$

Finally, the space of characteristic monomials for σ is just the direct sum of the spaces of characteristic monomials induced by the various partitions S, so that

$$\mathfrak{pfb}^!_{n,\sigma}(z) = \sum_S \chi_S$$

$\square$

5 A Koszul Formula for Graded Characters

In this section we will state and prove a generalization of the well-known Koszul formula which applies to quadratic algebras which have the 'Koszul' property. We briefly recall the necessary concepts. Our presentation follows [PP], to which the reader may refer for further information.

We assume given a quadratic algebra A, defined as in Subsection 2.2, by $A := TV/\langle R\rangle$. For each $n = 2, 3, \ldots$, and for $1 \le i \le n-1$, define $X_i^n := V^{\otimes i-1} \otimes R \otimes V^{\otimes n-1-i}$.

One can define a graded complex, known as the Koszul complex, whose degree n component is the following:

$$0 \longrightarrow X_1^n \cap \cdots \cap X_{n-1}^n \xrightarrow{d_1} X_2^n \cap \cdots \cap X_{n-1}^n \xrightarrow{d_2} \frac{X_3^n \cap \cdots \cap X_{n-1}^n}{X_1^n} \xrightarrow{d_3} \cdots$$
$$\cdots \xrightarrow{d_{i-1}} \frac{X_i^n \cap \cdots \cap X_{n-1}^n}{X_1^n + \cdots + X_{i-2}^n} \xrightarrow{d_i} \cdots$$
$$\xrightarrow{d_{n-2}} \frac{X_{n-1}^n}{X_1^n + \cdots + X_{n-3}^n} \xrightarrow{d_{n-1}} \frac{V^{\otimes n}}{X_1^n + \cdots + X_{n-2}^n} \xrightarrow{d_n} \frac{V^{\otimes n}}{X_1^n + \cdots + X_{n-1}^n} \longrightarrow 0$$

where we write U/V for $U/(U \cap V)$.

The map d_i is the composition of the obvious inclusion and projection:

$$\frac{X_i^n \cap \cdots \cap X_{n-1}^n}{X_1^n + \cdots + X_{i-2}^n} \hookrightarrow \frac{X_{i+1}^n \cap \cdots \cap X_{n-1}^n}{X_1^n + \cdots + X_{i-2}^n} \twoheadrightarrow \frac{X_{i+1}^n \cap \cdots \cap X_{n-1}^n}{X_1^n + \cdots + X_{i-1}^n}$$

With these d_i it is easy to check that the previous sequence is a complex, for each n.

The algebra A is said to be Koszul when the Koszul complex is exact for all $n \ge 2$. For A Koszul, one can show[13] that

$$A(z)A^!(-z) = 1 \tag{8}$$

where $A(z)$ is the Hilbert series encoding the dimensions of the graded components of A (and similarly for $A^!(z)$).

The Koszul formula (8) has the following generalization:

Theorem 4. *Let G be a finite group, let V be a finite-dimensional representation of G, and let G act diagonally on the (rational) tensor algebra TV. Let $R \subseteq V \otimes V$ be a submodule and suppose $A := TV/\langle R\rangle$ is a Koszul algebra.*

Then A is a graded representation of G whose character satisfies the 'Koszul' formula

$$A_\sigma(z)A_\sigma^!(-z) = 1 \tag{9}$$

where $A_\sigma(z)$ is the (graded) character of the representation A evaluated at the element $\sigma \in G$ (and similarly for $A_\sigma^!(z)$).

The usual Koszul formula (8) is just the case where $\sigma = 1$, the identity of G.

[13] See, for instance, [PP], Cor. 2.2.2.

Proof. Since R, and hence $\langle R \rangle$, is a submodule of TV, the fact that A is a G-module is clear. Moreover, since V and R are G-modules, so are the X_i^n, as well as their various intersections, sums and quotients, such as:

$$X_i^n \cap \cdots \cap X_{n-1}^n$$
$$X_1^n + \cdots + X_{i-2}^n$$
$$E_i := \frac{X_i^n \cap \cdots \cap X_{n-1}^n}{X_1^n + \cdots + X_{i-2}^n}$$

The kernel of d_i is (by exactness of the Koszul complex) the subspace:

$$F_i := \frac{X_i^n \cap \cdots \cap X_{n-1}^n}{X_1^n + \cdots + X_{i-1}^n} \hookrightarrow E_{i+1}$$

where we again write U/V for $U/(U \cap V)$.

By the discussion above, F_i is in in fact a submodule of E_{i+1}. Hence, by Maschke's theorem, there is a submodule $F_{i+1} \subseteq E_{i+1}$ such that:

$$E_{i+1}/F_i \cong F_{i+1} \text{ and } E_{i+1} = F_i \oplus F_{i+1}$$

as G-modules.

Hence the Koszul complex is isomorphic to the sequence of modules:

$$0 \to F_1 \to F_1 \oplus F_2 \to F_2 \oplus F_3 \to \ldots \to F_{n-1} \oplus F_n \to F_n \to 0$$

If we write χ_i for the character of F_i evaluated at σ, and $\chi_0 = \chi_{n+1} = 0$, it is clear that

$$\sum_{i=0}^{n} (-1)^i (\chi_i + \chi_{i+1}) = 0$$

But, by the additivity of characters of direct sums of modules, $(\chi_i + \chi_{i+1})$ is the character of $E_{i+1} = F_i \oplus F_{i+1}$. Moreover, one knows that

$$E_{i+1} = A^{!n-i} \otimes A^i$$

(this can be seen by inspection, but see also [PP], Prop. 1.6.2 and Prop. 2.3.1).

Hence, by the multiplicativity of characters of tensor products of modules, $(\chi_i + \chi_{i+1}) = A_\sigma^{!n-i} A_\sigma^i$ (A_σ^i is the character of the representation A^i evaluated at σ, and similarly for $A_\sigma^{!(n-i)}$). So we find that:

$$\sum_{i=0}^{n} (-1)^i A_\sigma^{!n-i} A_\sigma^i = 0 \tag{10}$$

for $n \geq 2$. In fact, the same equation (10) clearly holds also for $n = 0, 1$ (the case $n = 0$ corresponding to the trivial representation $A^0 = A^{!0} = \mathbb{Q}$). Since equation (10) is just the degree n part of equation (9), the result follows.

□

6 Final Comments

One can also ask for Hilbert series describing the decomposition of the graded algebras $\mathfrak{pvb}_n^!$ and $\mathfrak{pvb}_n$, and $\mathfrak{pfb}_n^!$ and $\mathfrak{pfb}_n$, into the irreducible representations of S_n. Of course, given the graded characters above, one can in principle determine this decomposition. However, in practice, and for arbitrary n, there is still a fair bit of work to be done – one approach might be to write down all the characters for all the conjugacy classes in S_n, determine the characters for the irreducibles (for instance using the Frobenius formula) and then solve for the decomposition. It would be much better to have Hilbert series directly expressing these decompositions.

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